Exercise 8

Solve the differential equation.

$$\frac{d^2y}{dx^2} + 4y = \sin 2x$$

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as the sum of a complementary solution and a particular solution.

$$y = y_c + y_p$$

The complementary solution satisfies the associated homogeneous equation.

$$\frac{d^2y_c}{dx^2} + 4y_c = 0 \tag{1}$$

This is a linear homogeneous ODE with constant coefficients, so it has solutions of the form $y_c = e^{rx}$.

$$y_c = e^{rx} \quad \to \quad \frac{dy_c}{dx} = re^{rx} \quad \to \quad \frac{d^2y_c}{dx^2} = r^2e^{rx}$$

Substitute these formulas into the ODE.

$$r^2e^{rx} + 4(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 + 4 = 0$$

Solve for r.

$$r = \{-2i, 2i\}$$

Two solutions to the ODE are e^{-2ix} and e^{2ix}. According to the principle of superposition, the general solution to equation (1) is a linear combination of these two.

$$\begin{aligned} y_c(x) &= C_1e^{-2ix} + C_2e^{2ix} \\ &= C_1(\cos 2x - i\sin 2x) + C_2(\cos 2x + i\sin 2x) \\ &= (C_1 + C_2)\cos 2x + (-iC_1 + iC_2)\sin 2x \\ &= C_3\cos 2x + C_4\sin 2x \end{aligned}$$

C_3 and C_4 are arbitrary constants. On the other hand, the particular solution satisfies the original ODE.

$$\frac{d^2y_p}{dx^2} + 4y_p = \sin 2x \tag{3}$$

Since the inhomogeneous term is a sine, the particular solution would be $y_p = A\cos 2x + B\sin 2x$. $\sin 2x$ is already part of y_c, though, so an extra factor of x is needed: $y_p = x(A\cos 2x + B\sin 2x)$.

$$\begin{aligned} y_p &= x(A\cos 2x + B\sin 2x) \\ \frac{dy_p}{dx} &= (A\cos 2x + B\sin 2x) + x(-2A\sin 2x + 2B\cos 2x) \\ \frac{d^2y_p}{dx^2} &= (-2A\sin 2x + 2B\cos 2x) + (-2A\sin 2x + 2B\cos 2x) + x(-4A\cos 2x - 4B\sin 2x) \end{aligned}$$

Substitute these formulas into equation (3).

$$[(-2A\sin 2x + 2B\cos 2x) + (-2A\sin 2x + 2B\cos 2x) + x(-4A\cos 2x - 4B\sin 2x)] + 4[x(A\cos 2x + B\sin 2x)] = \sin 2x$$

Simplify the left side.

$$4B\cos 2x + (-4A)\sin 2x = \sin 2x$$

Match the coefficients to get a system of equations for A and B.

$$4B = 0$$

$$-4A = 1$$

Solving it yields

$$A = -\frac{1}{4} \quad \text{and} \quad B = 0.$$

The particular solution is then

$$\begin{aligned} y_p &= x(A\cos 2x + B\sin 2x) \\ &= -\frac{1}{4}x\cos 2x. \end{aligned}$$

Therefore, the general solution to the original ODE is

$$\begin{aligned} y &= y_c + y_p \\ &= C_3\cos 2x + C_4\sin 2x - \frac{1}{4}x\cos 2x. \end{aligned}$$